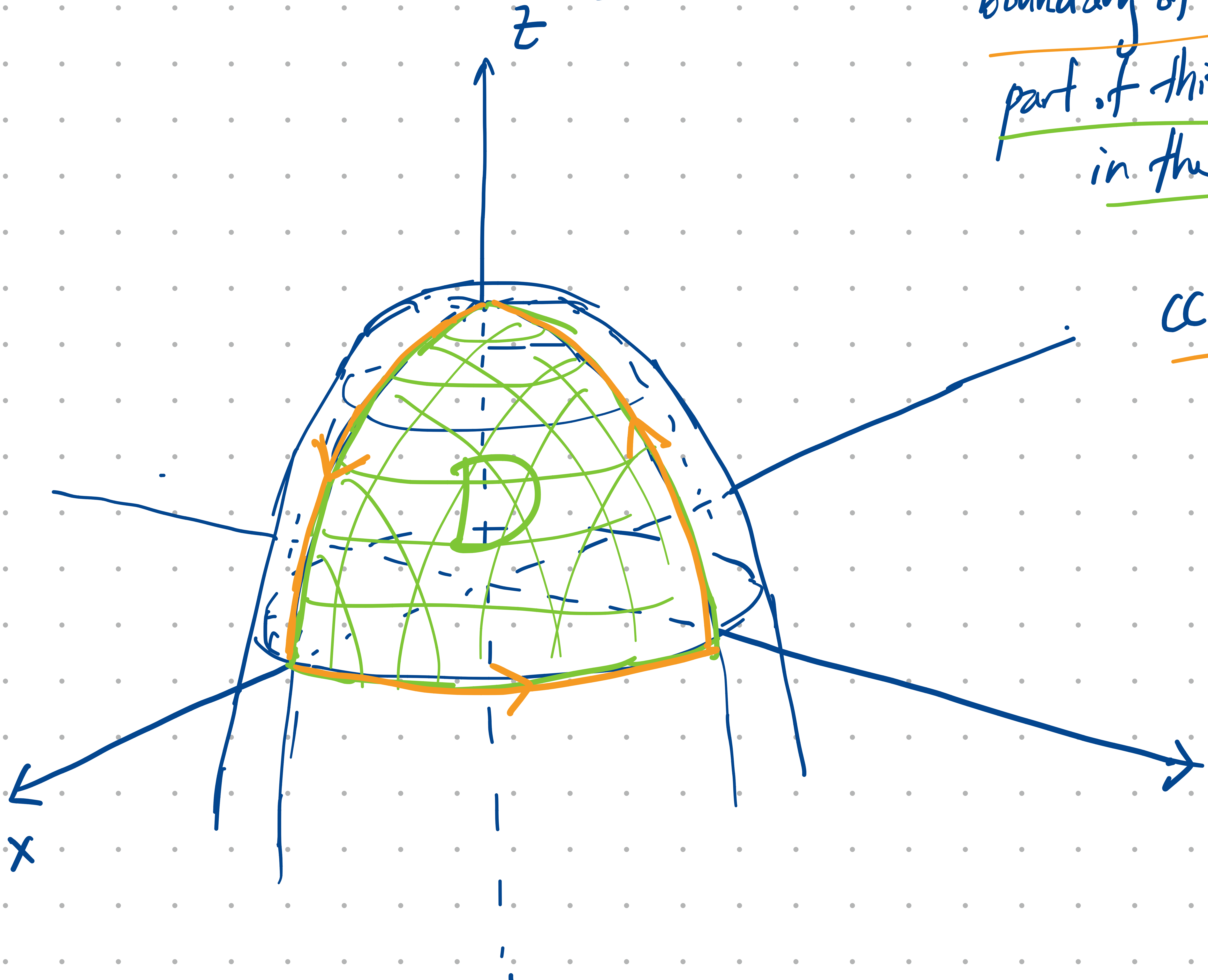
16.8 #9

Evaluate $\int_C \vec{F} \cdot d\vec{r}$ where C is the boundary of the part of this paraboloid in the first octant, CCW from above.



$\vec{F} = \langle xy, yz, zx \rangle$

One could tackle this directly using §16.2 methods by breaking C into three parts and computing those line integrals directly.

But since we already have an obvious surface whose boundary is C, Stokes seems like a better choice

D

C = Boundary of D ccw from above

(RHR)

D oriented upwards

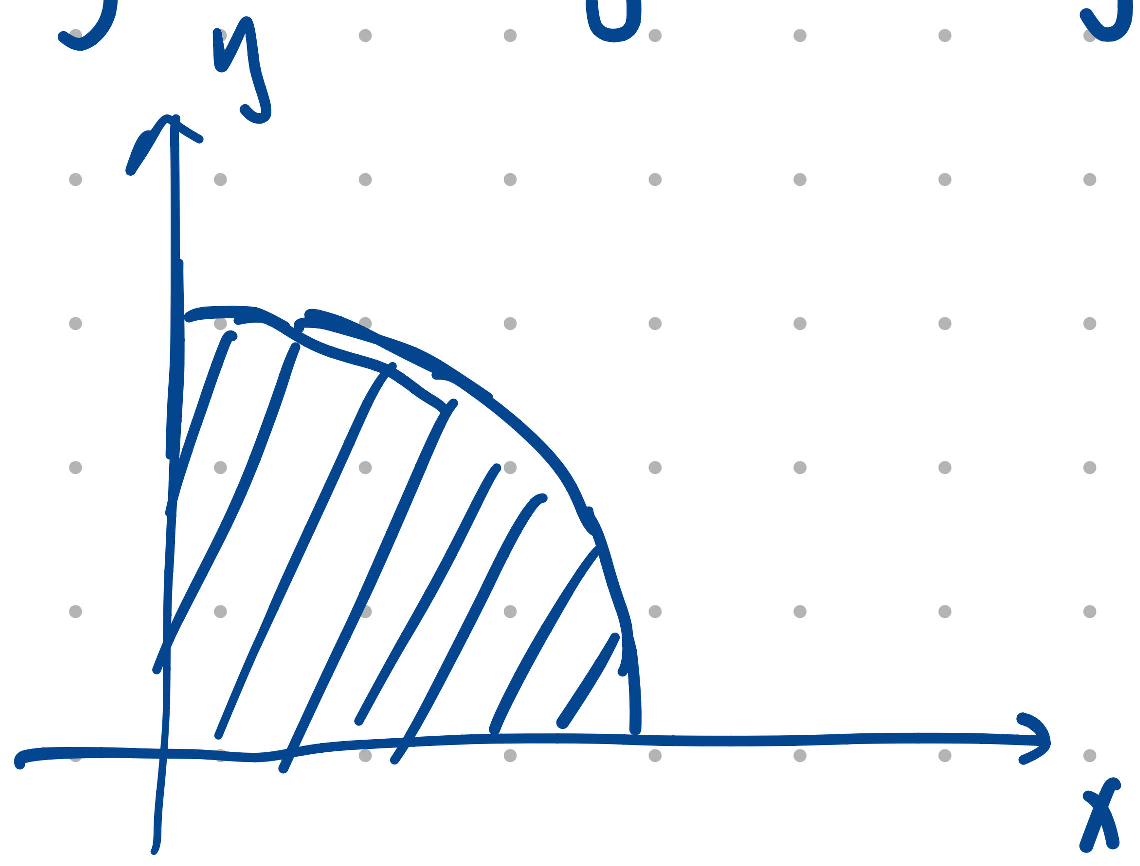
$$\int_C \vec{F} \cdot d\vec{r} = \iint_D (\nabla \times \vec{F}) \cdot d\vec{S}$$

Now need to write out the integral...

.. need to parametrize D.

Method 1:

$\vec{r}(x,y) = \langle x, y, 1-x^2-y^2 \rangle$

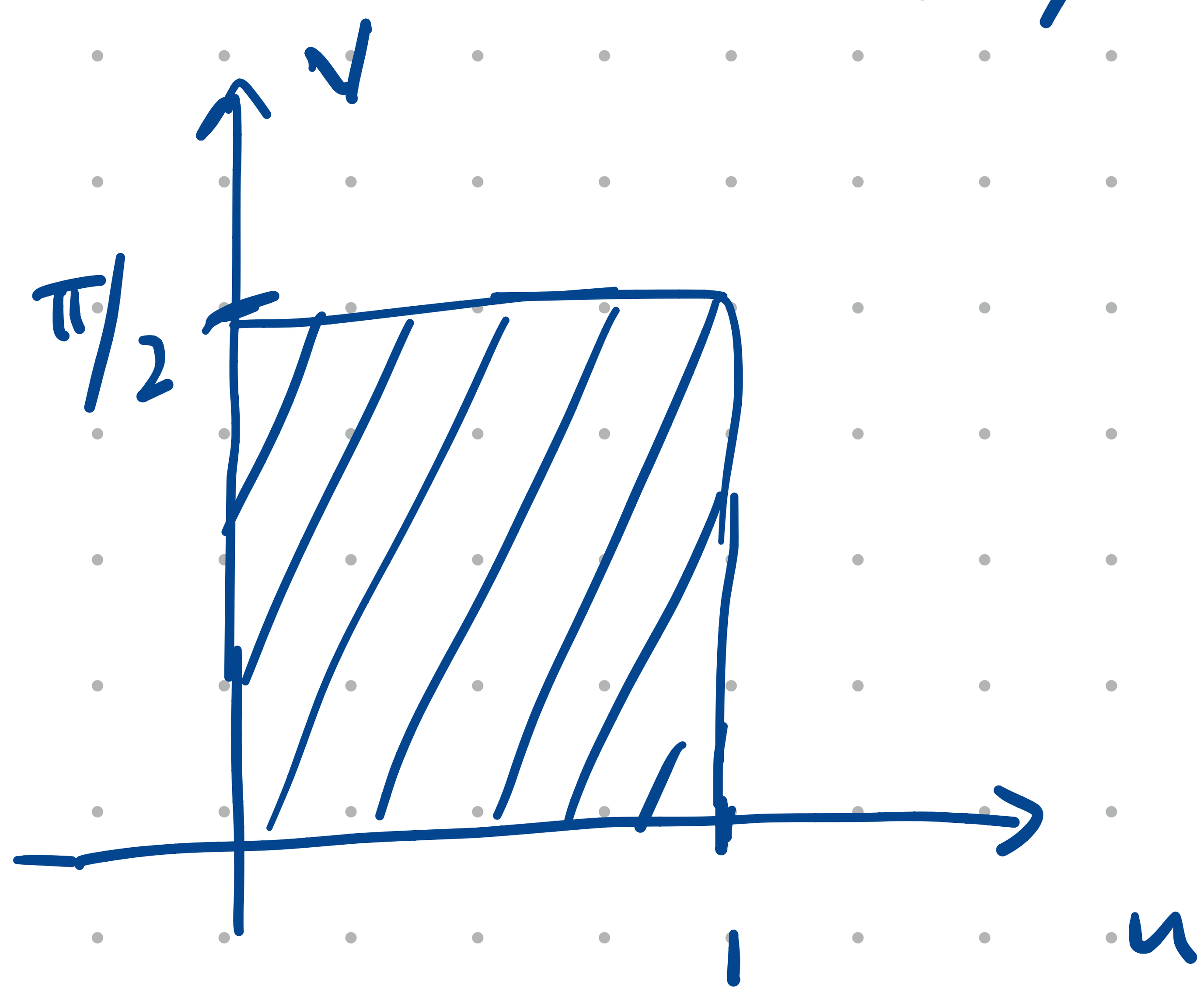


After setting up this integral, will probably want to switch to polar.

....

Method 2:

$\vec{r}(u,v) = \langle u\cos v, u\sin v, 1-u^2 \rangle$



Doing it this way combines the two steps of method 1 into one step.

$$\iint_D (\nabla \times \vec{F}) \cdot d\vec{S} = \int_0^{\pi/2} \int_0^1 (\nabla \times \vec{F}) \cdot \left(\overset{?}{\pm}\, \vec{r}_u \times \vec{r}_v \right) du\, dv$$

which one do I take?

Let's compute $\vec{r}_u \times \vec{r}_v$:

$$\vec{r}_u = \langle \cos v,\ \sin v,\ -2u \rangle$$

$$\vec{r}_v = \langle -u\sin v,\ u\cos v,\ 0 \rangle$$

$$\vec{r}_u \times \vec{r}_v = \langle 2u^2\cos v,\ 2u^2\sin v,\ u \rangle$$

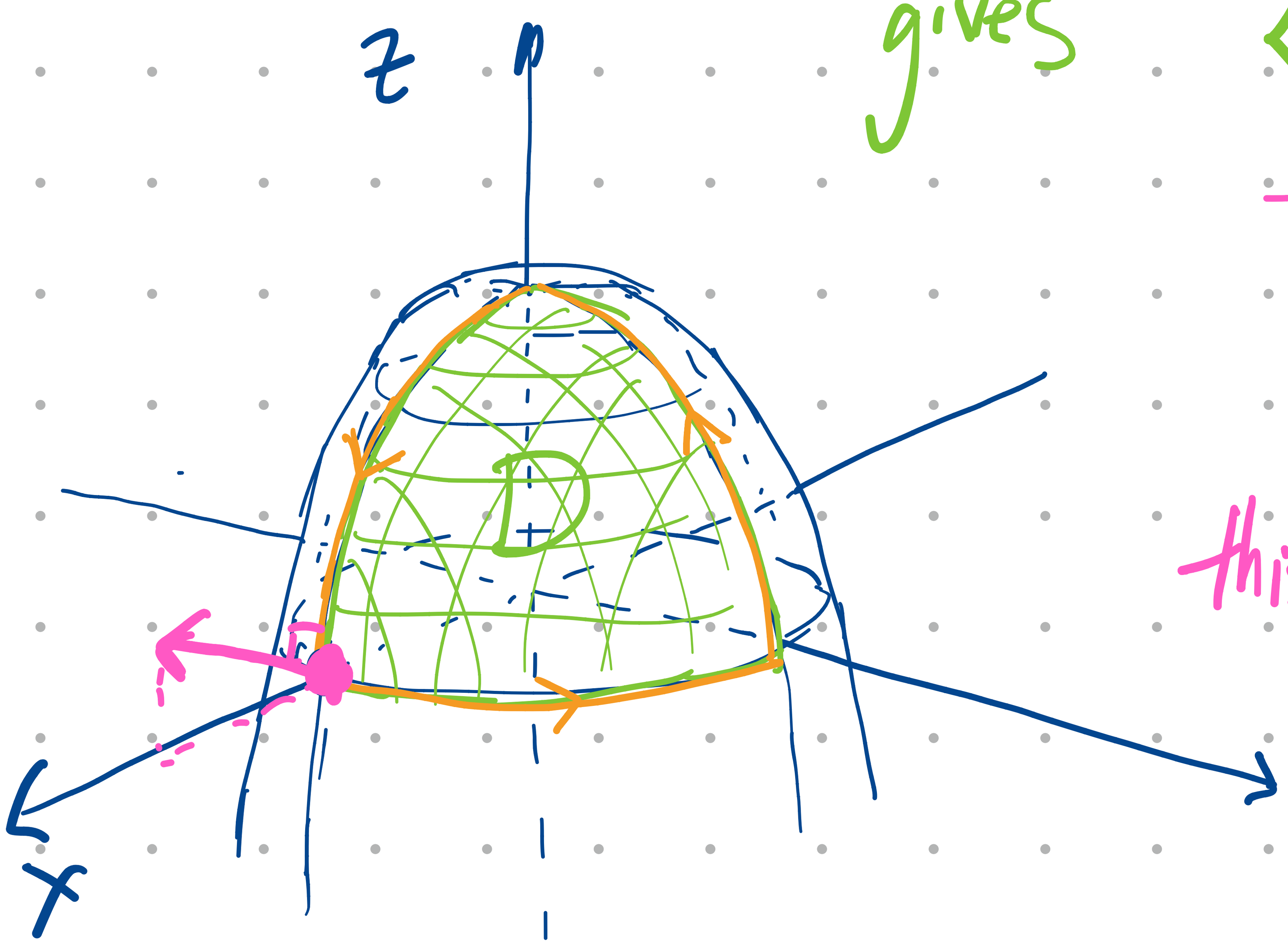
Question: Is this the vector I want? ✓
(or do I want its negative?)

yes b/c for example $u \geq 0$ so this is pointing up.

Alternatively: plug in parameter values (that don't give $\vec{0}$)

e.g. $u=1 \quad v=0$

gives $\langle 2, 0, 1 \rangle$

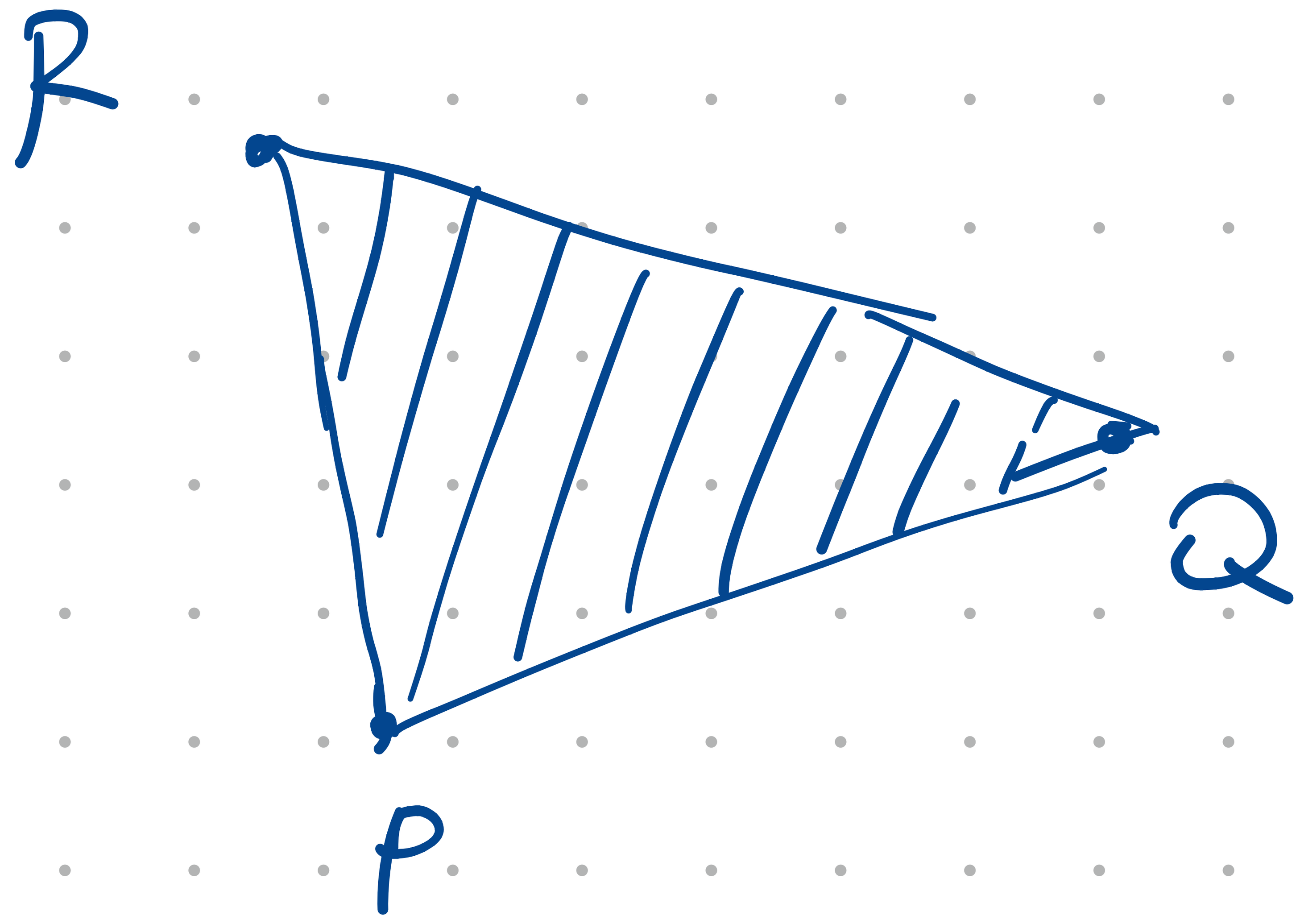


this looks good ✓

Compute $\nabla \times \vec{F}$, substitute in terms of u, v,
then the integral just becomes a Ch 15 problem.

Problem Consider the triangle w/ vertices

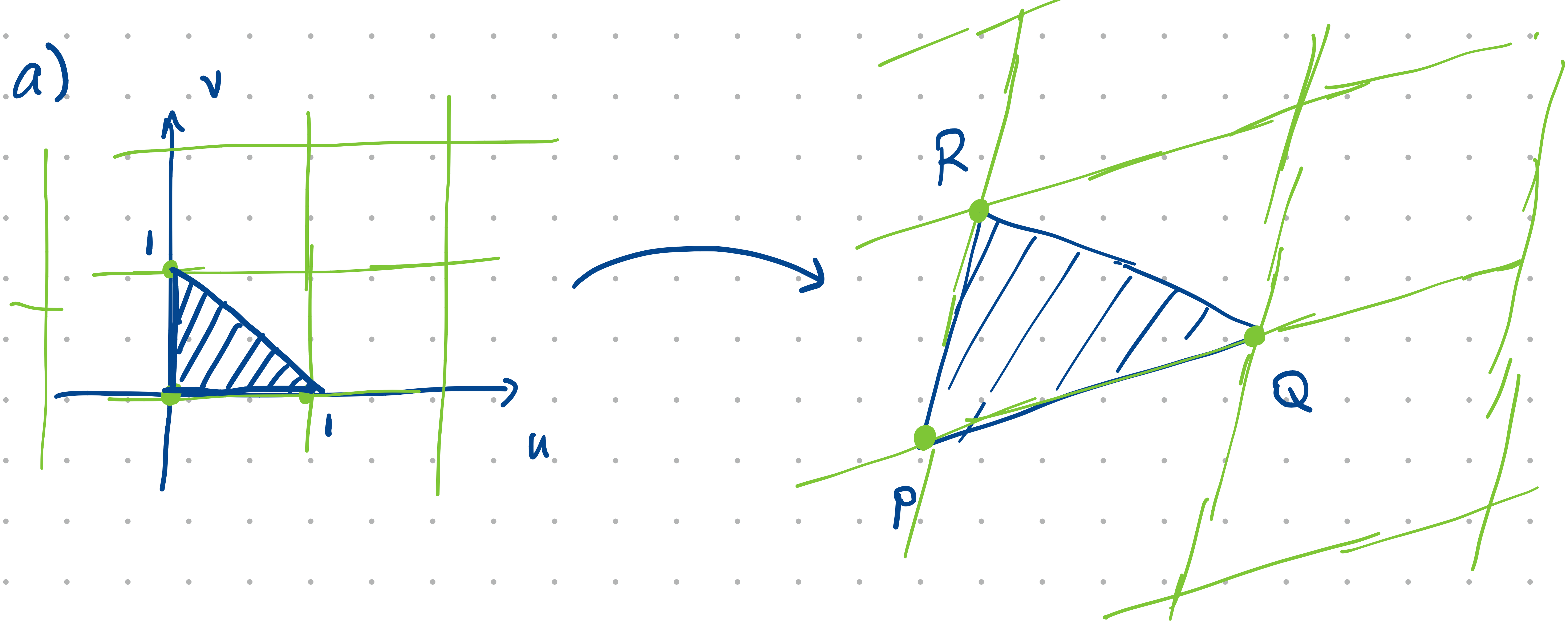
$P(1,0,1)$, $Q(-2,1,3)$, $R(4,2,5)$



a) Parametrize the solid triangle

b) Compute the flux of $\langle x, y, z \rangle$ through the triangle in the direction away from the origin.

a)



$$\vec{r}(u,v) = \overrightarrow{OP} + u\,\overrightarrow{PQ} + v\,\overrightarrow{PR}$$

$$\overrightarrow{OP} = \langle 1,0,1\rangle$$

$$= \langle 1,0,1\rangle + u\langle -3, 1, 2\rangle + v\langle 3,2,4\rangle$$

$$= \langle 1-3u+3v,\ u+2v,\ 1+2u+4v\rangle$$

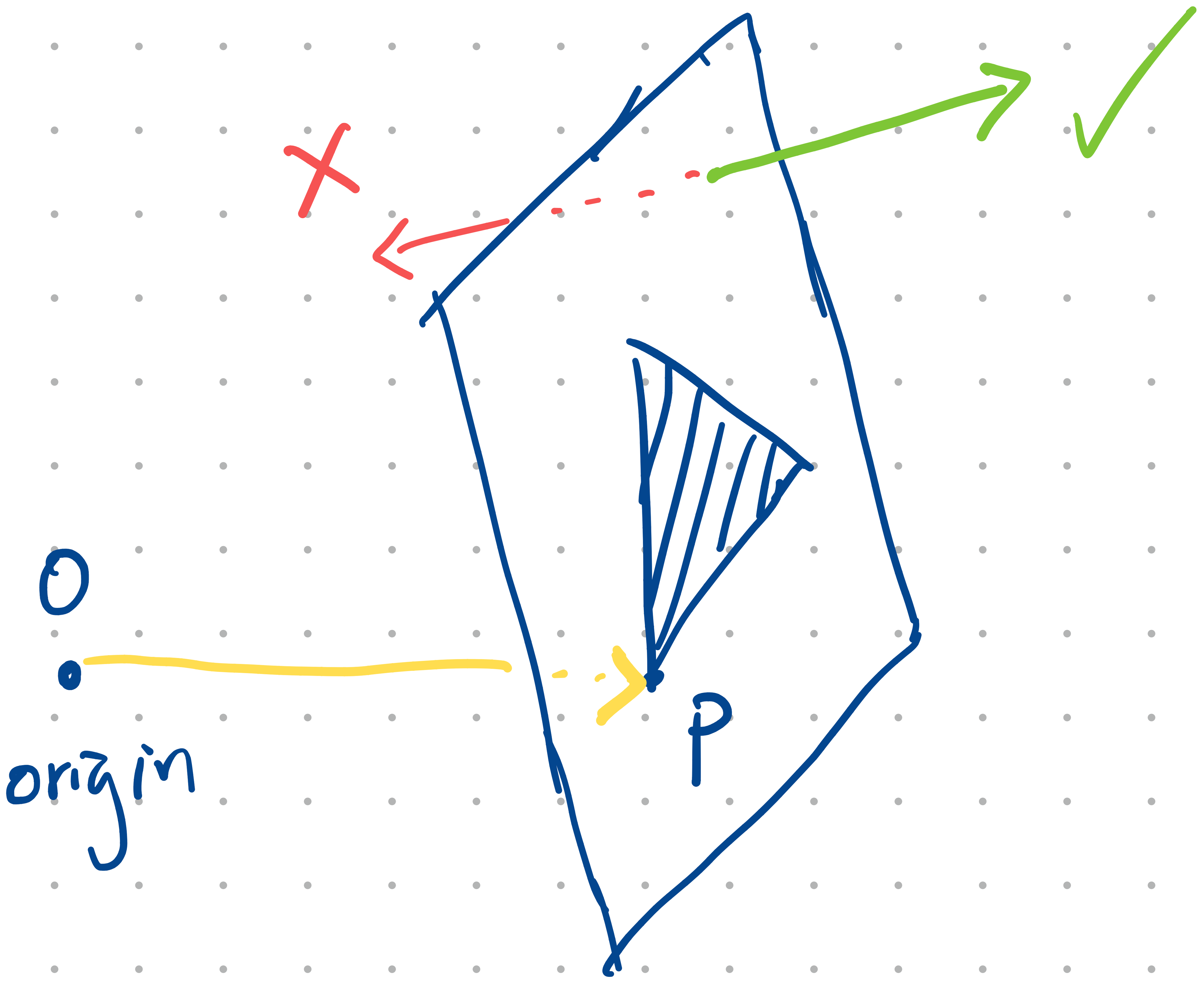
parametrization

$\vec{r}_u = \langle -3,1,2\rangle$

$\vec{r}_v = \langle 3,2,4\rangle$

$$\vec{r}_u \times \vec{r}_v = \langle 0, 18, -9\rangle$$

Q: Is this pointing away from origin?



To check: pick a point on the plane, e.g. P

$$\overrightarrow{OP} \cdot (\vec{r}_u \times \vec{r}_v) = \langle 1, 0, 1 \rangle \cdot \langle 0, 18, -9 \rangle$$

$$= -9.$$ so

$\langle 0, 18, -9 \rangle$ is pointing in the wrong way.

So we use $\langle 0, -18, 9 \rangle$ instead.

$$\int_0^1 \int_0^{1-u} \langle 1-3u+3v, u+2v, 1+2u+4v \rangle \cdot \langle 0, -18, 9 \rangle \, dv\, du$$

Food for thought: Can you answer (b) using the Divergence Thm?